

Proper time in Weyl space-time

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Abstract

We discuss the question of whether or not a general Weyl structure is a suitable mathematical model of space-time. This is an issue that has been in debate since Weyl formulated his unified field theory for the first time. We do not present the discussion from the point of view of a particular unification theory, but instead from a more general standpoint, in which the viability of such a structure as a model of space-time is investigated. Our starting point is the well known axiomatic approach to space-time given by Elhers, Pirani and Schild (EPS). In this framework, we carry out an exhaustive analysis of what is required for a consistent definition for proper time and show that such a definition leads to the prediction of the so-called "second clock effect". We take the view that if, based on experience, we were to reject space-time models predicting this effect, this could be incorporated as the last axiom in the EPS approach. Finally, we provide a proof that, in this case, we are led to a Weyl integrable space-time (WIST) as the most general structure that would be suitable to model space-time.

I. INTRODUCTION

The axiomatic approach to space-time proposed by Ehlers, Pirani and Schild in [1], tries to build a suitable mathematical model of space-time from basic assumptions about the behavior of freely falling particles and the propagation of light rays. These are considered to be the *primitive concepts* and other constructions should be derived from these elements plus some set of hypotheses about their behavior which should be as natural as possible. This program led EPS to show that the propagation of light determines a *conformal structure* on space-time, while the motion of freely falling particles determines a *projective structure* on space-time. They require these two structures to satisfy a *compatibility condition*, which then leads to a *Weyl structure* as the resulting mathematical model. Up to this point, they conclude that the propagation of light and the motion of freely falling particles determine a Weyl structure as a natural model of space-time. They proceed to propose two (apparently different) possible axioms which then reduce the former to the structure of a *Weyl integrable* space-time¹. The previous axioms express quite general ideas which deal with simple aspects of the motion of freely falling particles and light rays, while the last axiom seems to deal with much more delicate details regarding the behavior of *clocks*. In particular, in order to discuss the behavior of clocks in this context, we should first have a well-defined and physically sensible notion of *proper time*. This notion seems to be at the core of the different discussions, which took place after Weyl's proposal of his unified theory, concerning the viability of Weyl's space-time as an acceptable model for physics. Some of these discussions, which involved Weyl, Einstein, Eddington and Pauli, among others, can be reviewed in [2]. Although in their paper EPS give a mathematically well-defined notion of proper time, this notion, in the way they introduce it, does not seem to be motivated by the motion of freely falling particles or light rays [1]. On the other hand, the existence of a well-defined and physically sensible definition of proper time in a Weyl structure has been carefully discussed by V. Perlick [3]. At first sight, Perlick's definition might not seem to be very much related with the one given by EPS. In this paper, we will show that the concept of Perlick's proper

¹ Actually, after showing that Weyl's curvature F has to vanish, they conclude that space-time geometry has a Riemannian representation. This is true, since the vanishing of F implies (in a simply connected domain) that the Weyl structure is integrable, and in any Weyl integrable structure there is a Riemannian representative of the class.

time may be motivated by the axiomatic approach given by EPS, and that his definition is not only mathematically well-defined, but also physically sensible. We then will show the equivalence of EPS's and Perlick's definition, and that the latter leads to the second clock effect. Finally, we will show that, if we were to rule out models of space-time exhibiting such an effect, then we would arrive at what is called a "Weyl integrable space-time" structure. In fact, this claim has been stated previously, using, however, a different line of reasoning [4].

II. WEYL STRUCTURES

Since in this paper we will accept, in view of [1], that the motion of freely falling particles and light rays determine a Weyl structure on space-time, we will shortly review the basic definitions of such a structure.

A Weyl manifold is a triple (M, g, ω) , where M is a differentiable manifold, g a semi-Riemannian metric on M , and ω a 1-form field on M . We assume that M is endowed with a torsion-free linear connection ∇ satisfying the following *compatibility condition*:

$$\nabla g = g \otimes \omega. \quad (1)$$

Results concerning the existence and uniqueness of such a connection are straightforward and their proofs are analogous to those known for the Riemannian case [5]. It can easily be shown that, in local coordinates, the components of the Weyl connection ∇ have form:

$$\Gamma_{ac}^u = \frac{1}{2}g^{bu}(\partial_a g_{bc} + \partial_c g_{ab} - \partial_b g_{ca}) + \frac{1}{2}g^{bu}(\omega_b g_{ca} - \omega_a g_{bc} - \omega_c g_{ab}).$$

An important and useful result concerning Weyl connections is given by the following proposition:

Proposition 1. *A linear connection ∇ is said to be compatible with a Weyl structure (M, g, ω) if, and only if, $\forall X, Y, Z \in \mathfrak{X}(M)$ we have*

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z) + \omega(X)g(Y, Z). \quad (2)$$

Let us now note that the compatibility condition (1) is invariant under the following group of transformations:

$$\begin{cases} \bar{g} = e^{-f} g \\ \bar{\omega} = \omega - df \end{cases} \quad (3)$$

where f is an arbitrary smooth function defined on M . By this we mean that if ∇ is compatible with (M, g, ω) , then it is also compatible with $(M, \bar{g}, \bar{\omega})$. It is easy to check that these transformations define an equivalence relation between Weyl manifolds. In this way, every member of the class is compatible with the same connection, hence has the same geodesics, curvature tensor and any other property that depends only on the connection. This is the reason why it is regarded more natural, when dealing with Weyl manifolds, to consider the whole class of equivalence $(M, [g], [\omega])$ rather than working with a particular element of this class. In this sense, it is argued that only geometrical quantities that are invariant under (3) are of real significance in the case of Weyl geometry. Following the same line of argument, it is also assumed that only physical theories and physical quantities presenting this kind of invariance should be considered of interest in this context. To conclude this section, we remark that when the one-form field ω is an exact form, then the Weyl structure is called *integrable*.

III. PROPER TIME

In this section, the main idea is to introduce a definition of *proper time* which would naturally fit the axiomatic approach proposed by Ehlers, Pirani and Schild. In this work, we will disregard the last of the EPS² axioms, however retaining that the motion of freely falling particles and light rays determine a Weyl structure as a suitable mathematical model of space-time. In this way we will think of the space-time geometry as given by a structure of the form $(M, [g], [\omega])$. On the other hand, in this scenario, we will use the following idea

² In [1], in order to reduce the Weyl structure to a Riemannian one, two possible additional axioms, regarding the behaviour of clocks, are introduced. The first one is stated as follows. Given two freely falling, infinitesimally proximate clocks C_1 and C_2 , if we consider a regular sequence of events (p_1, p_2, \dots) in the world line of C_1 determined by the ticking of this clock, and the Einstein-simultaneous sequence of events (q_1, q_2, \dots) in the world line of C_2 , then (q_1, q_2, \dots) should also be a regular sequence of events. With the help of the geodesic deviation equation, EPS show that this hypothesis reduces the Weyl structure to a Riemannian one. Another way to achieve the same the same goal, is the consider as an axiom that the "norm" of parallel transported vector fields at a point can not depend on the curve chosen to make the transport, associating this "norm", for the case of time-like curves, to the ticking rate of clocks.

to guide an axiomatic definition of proper time:

A freely falling particle should see itself freely falling.

With this statement we mean that the proper time parametrization of a freely falling particle should be such that the equation of motion of this particle satisfies the geodesic equation. This idea has already been suggested in [6]. A freely falling particle $\gamma : I \mapsto M$, $u \mapsto \gamma(u)$, is freely falling if it is a pregeodesic, that is, if

$$\frac{D\gamma'(u)}{du} = f(\gamma(u))\gamma'(u). \quad (4)$$

Our principle is that proper time parametrization is a reparametrization of γ that transforms it into a geodesic. Then, if $\gamma(\tau)$ is a time-like curve representing a freely falling particle, it is straightforward to see that γ is parametrized by proper time if, and only if

$$g(\gamma'(\tau), \frac{D\gamma'(\tau)}{d\tau}) = 0. \quad (5)$$

This notion of proper time fits naturally into the picture proposed by EPS, since we are using the basic concepts of their construction to motivate it. Our next step should be to generalize this definition for arbitrary time-like curves, which represent arbitrary particles. Obviously, since, in general, particles are not freely falling, (4) does not generalize naturally to the general case. However, (5) does, and, as we have just seen, (4) and (5) are equivalent in the case of freely falling particles. This leads us to the following definition, which is precisely the one given by V. Perlick in [3]:

Definition 1. A time-like curve $\gamma : I \mapsto M$, $u \mapsto \gamma(u)$, is called a *standard clock* if $\frac{D\gamma'}{du}$ is orthogonal to $\gamma'(u)$.

In order to check that this definition is mathematically consistent with the fact that we are working with a Weyl structure, we should check that it is independent of the representative member of the class chosen to carry out the computations. Suppose that for some particular $g \in [g]$ we have $g(\gamma', \frac{D\gamma'}{du}) = 0$. Then, since ∇ is independent of the choice of the representative, so is the covariant derivative, and hence $\frac{D\gamma'}{du}$ does not depend on this choice. Also, since any other $\tilde{g} \in [g]$ is related to g by a conformal transformation, then orthogonality of vectors is preserved. Thus, the definition is consistent in the whole class.

Also, in order for this definition to be sensible, we should show that any time-like curve can be parametrized by this kind of parametrization. To see this, consider a time-like curve

$$\begin{aligned}\gamma : I &\mapsto M \\ t &\mapsto \gamma(t)\end{aligned}$$

We are looking for a reparametrization of γ such that the reparametrized curve is a standard clock. This means that we are looking for a diffeomorphism

$$\begin{aligned}\mu : I &\mapsto I' \\ t &\mapsto \tau\end{aligned}$$

such that

$$\begin{aligned}\tilde{\gamma} &= \gamma \circ \mu^{-1} : I' \mapsto M \\ \tau &\mapsto \gamma(\mu^{-1}(\tau))\end{aligned}$$

is a standard clock. With this set-up, we see that $\gamma = \tilde{\gamma} \circ \mu : I \mapsto M$ and

$$\frac{D\gamma'(t)}{dt} = \frac{d^2\mu}{dt^2}\tilde{\gamma}'(\mu(t)) + \left(\frac{d\mu}{dt}\right)^2 \frac{D\tilde{\gamma}'}{d\tau}(\mu(t)). \quad (6)$$

Also from (6) we get

$$\begin{aligned}g\left(\frac{D\gamma'(t)}{dt}, \gamma'(t)\right) &= \frac{d^2\mu}{dt^2}g(\tilde{\gamma}'(\mu(t)), \gamma'(t)) + \left(\frac{d\mu}{dt}\right)^2 g\left(\frac{D\tilde{\gamma}'}{d\tau}(\mu(t)), \gamma'(t)\right) \\ &= \frac{1}{\frac{d\mu}{dt}} \frac{d^2\mu}{dt^2} g(\gamma'(t), \gamma'(t)) + \left(\frac{d\mu}{dt}\right)^3 g\left(\frac{D\tilde{\gamma}'}{d\tau}(\mu(t)), \tilde{\gamma}'(\mu(t))\right)\end{aligned}$$

Then, the following equation is satisfied:

$$\frac{d^2\mu}{dt^2} - \frac{g(\gamma'(t), \frac{D\gamma'(t)}{dt})}{g(\gamma'(t), \gamma'(t))} \frac{d\mu}{dt} + \left(\frac{d\mu}{dt}\right)^4 \frac{g(\frac{D\tilde{\gamma}'}{d\tau}(\mu(t)), \tilde{\gamma}'(\mu(t)))}{g(\gamma'(t), \gamma'(t))} = 0.$$

From this last equation we see that $\tilde{\gamma}$ is a standard clock if, and only if, the reparametrization μ satisfies the following differential equation:

$$\frac{d^2\mu}{dt^2} - \frac{g(\gamma'(t), \frac{D\gamma'(t)}{dt})}{g(\gamma'(t), \gamma'(t))} \frac{d\mu}{dt} = 0 \quad (7)$$

We then see that if a reparametrization μ^{-1} makes $\gamma \circ \mu^{-1}$ a standard clock, then it satisfies (7). Conversely, given a solution of (7) with initial conditions such that $\frac{d\mu}{dt}(t_0) \neq 0$,

then there is a neighborhood I of t_0 where $\mu : I \mapsto \mu(I) = I'$ is a diffeomorphism, and hence the reparametrization $\gamma \circ \mu^{-1} : I' \mapsto M$ will be a standard clock. Thus, we can state that given a time-like curve $\gamma(t)$, there is a reparametrization which makes it a standard clock if, and only if, the equation (7) admits a solution with $\frac{d\mu}{dt}(t_0) \neq 0$. Since this type of differential equation always admits solutions for given initial data $\mu(t_0), \mu'(t_0)$, then a given time-like curve γ can always be reparametrized in a neighborhood of any point, so as to make it a standard clock. Seeing that this definition makes mathematical sense, we will use it to define *proper time*.

Definition 2. We will say that a time-like curve γ is parametrized by proper time if the parametrized curve is a standard clock.

Concerning the above definitions, the following remarks should be made:

- 1) Using the same notations as above, the map $\mu : I \mapsto I'$, maps an arbitrary parametrization t to proper time $\tau = \mu(t)$.
- 2) Since (7) is linear, if μ is a solution then $\tilde{\mu}(t) = a\mu(t) + b$ is also a solution, where a, b are arbitrary constants. These constants are determined by the initial conditions, and they just represent the scale (a) and the zero of the clock (b).

It will be important to note that a general solution of equation (7) can be obtained. In order to do this, first note that from *Proposition 1* we have

$$\frac{d}{dt} \ln(-g(\gamma'(t), \gamma'(t))) = \frac{1}{g(\gamma'(t), \gamma'(t))} \left\{ 2g(\gamma'(t), \frac{D\gamma'(t)}{dt}) + \omega(\gamma'(t))g(\gamma'(t), \gamma'(t)) \right\}.$$

This leads to

$$\frac{g(\gamma'(t), \frac{D\gamma'(t)}{dt})}{g(\gamma'(t), \gamma'(t))} = \frac{1}{2} \left\{ \frac{d}{dt} \ln(-g(\gamma'(t), \gamma'(t))) - \omega(\gamma'(t)) \right\}. \quad (8)$$

Going back to (7) and using (8) we get

$$\frac{d^2\mu}{dt^2} - \frac{1}{2} \left\{ \frac{d}{dt} \ln(-g(\gamma'(t), \gamma'(t))) - \omega(\gamma'(t)) \right\} \frac{d\mu}{dt} = 0. \quad (9)$$

In order to integrate this equation, first define $\psi \doteq \frac{d\mu}{dt}$, and then (9) is reduced to a first order linear ordinary differential equation for ψ , which can easily be integrated, yielding

$$\frac{d\mu(t)}{dt} = \frac{d\mu(t_0)}{dt} \left[\frac{g(\gamma'(t), \gamma'(t))}{g(\gamma'(t_0), \gamma'(t_0))} \right]^{\frac{1}{2}} e^{-\frac{1}{2} \int_{t_0}^t \omega(\gamma'(u)) du}. \quad (10)$$

We can now integrate this equation once more and get the general solution for (7). It is not difficult to see that doing this we get the following:

$$\mu(t) = \frac{\frac{d\mu(t_0)}{dt}}{(-g(\gamma'(t_0), \gamma'(t_0)))^{\frac{1}{2}}} \int_{t_0}^t e^{-\frac{1}{2} \int_{t_0}^u \omega(\gamma'(s)) ds} (-g(\gamma'(u), \gamma'(u))^{\frac{1}{2}} du + \mu_0,$$

If $\Delta\tau(t)$ denotes the elapsed proper time between t_0 and t , we can write

$$\Delta\tau(t) = \frac{\frac{d\tau(t_0)}{dt}}{(-g(\gamma'(t_0), \gamma'(t_0)))^{\frac{1}{2}}} \int_{t_0}^t e^{-\frac{1}{2} \int_{t_0}^u \omega(\gamma'(s)) ds} (-g(\gamma'(u), \gamma'(u))^{\frac{1}{2}} du. \quad (11)$$

As a final comment about the mathematical consistency of our present definition of proper time, we remark that the expression (11), which is the expression to be used when computing the elapsed proper time measured by an observer between two events, is invariant both under the Weyl transformations (3) and reparametrizations of γ . Therefore, we can say that we have a mathematically consistent definition of proper time in the framework of Weyl geometry. Let us now analyze whether it is physically sensible to adopt this definition. In order to address this question we will consider three points.

i) The Riemannian limit

We claim that the above definition of proper time coincides with the usual definition adopted in general relativity and other metric theories of gravity where the underlying space-time structure is that of a Riemannian manifold. In order to see this, consider this definition when the manifold (M, g) is Riemannian. Let γ be a time-like curve. In this case, the compatibility of ∇ with g gives

$$g\left(\frac{D\gamma'}{dt}(t), \gamma'(t)\right) = \frac{1}{2} \frac{d}{dt} g(\gamma'(t), \gamma'(t)). \quad (12)$$

So $\gamma(t)$ is standard clock if and only if $g(\gamma'(t), \gamma'(t))$ is constant along $\gamma(t)$. This means that γ is parametrized by arc-length, or an affine reparametrization of it. We conclude that our definition is consistent with the Riemannian limit.

ii) The WIST limit

If the 1-form field ω is exact, i.e, if $\omega = d\phi$, where ϕ is some smooth scalar field ϕ on M , then we say that the Weyl structure is *integrable*, and accordingly the resulting space-time

is called *Weyl Integrable Space-Time (WIST)*. This kind of geometry has recently attracted the attention of some cosmologists [6],[7],[8]. In the case of WIST, it is already known that it is possible to define the proper time interval between two events along a curve $\gamma(t)$ in an invariant way as [8]

$$\Delta\tau = \int_{u_1}^{u_2} e^{-\frac{1}{2}\phi(\gamma(u))} \sqrt{g(\gamma'(u), \gamma'(u))} du. \quad (13)$$

It is easy to see that this quantity is invariant under the group of Weyl transformation defined in (3). We want to show that Perlick's definition reduces to this expression in the case of a WIST model. In order to do this, consider the explicit expression (11) and set $\omega = d\phi$. This will lead to

$$\begin{aligned} \Delta\tau &= \frac{\frac{d\tau(u_0)}{du}}{(-g(\gamma'(u_0), \gamma'(u_0))e^{-\phi(\gamma(u_1))})^{\frac{1}{2}}} \int_{u_1}^{u_2} e^{-\frac{1}{2}\phi(\gamma(u'))} (-g(\gamma'(u'), \gamma'(u'))^{\frac{1}{2}} du' \\ &= C \int_{u_1}^{u_2} e^{-\frac{1}{2}\phi(\gamma(u'))} (-g(\gamma'(u'), \gamma'(u'))^{\frac{1}{2}} du' \end{aligned}$$

which, by an appropriate setting of the scale to make $C = 1$, reproduces the WIST definition of proper time.

iii) Additivity

It seems that in any plausible physical definition proper time intervals should be additive. This means that if an observer experiences three events A, B and C in that order, then given two identical clocks, the time interval measured by a single clock from A to C should be the same as the sum of the time intervals measured by the other from A to B , and from B to C . Thus, for the above definition of proper time to be acceptable, it must be the case that if we use (11) to make computations in both situations, the results should be the same.³

In order to check that the definition for Perlick's proper time is additive, we need to compute the proper time elapsed between A and B , and between B and C , separately by using (11). Then we have to add these results, and this should equal the result we would get

³ We consider here the issue of additivity since there are other possible definitions for proper time in the context of Weyl's space-time that have Riemannian and WIST limits, but in which time is not additive. As an example of such a definition, consider $\Delta\tau$ given by $\Delta\tau(u) = \int_{u_0}^u e^{-\frac{1}{2} \int_{u_0}^{u'} \omega(\gamma'(s)) ds} (-g(\gamma'(u'), \gamma'(u'))^{\frac{1}{2}} du'$.

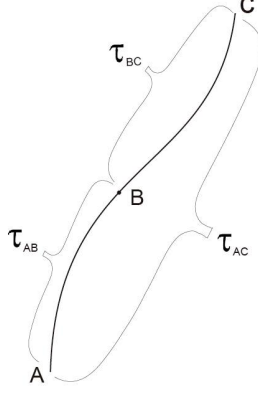


FIG. 1: Additivity of proper time.

if we were to make a single computation between A and C . Carrying out these calculations we get

$$\tau_{AB} = \frac{\tau'(t_A)}{(-g(\gamma'(t_A), \gamma'(t_A)))^{\frac{1}{2}}} \int_{t_A}^{t_B} e^{-\frac{1}{2} \int_{t_A}^t \omega(\gamma'(s)) ds} (-g(\gamma'(t), \gamma'(t)))^{\frac{1}{2}} dt$$

$$\tau_{BC} = \frac{\tau'(t_B)}{(-g(\gamma'(t_B), \gamma'(t_B)))^{\frac{1}{2}}} \int_{t_B}^{t_C} e^{-\frac{1}{2} \int_{t_B}^t \omega(\gamma'(s)) ds} (-g(\gamma'(t), \gamma'(t)))^{\frac{1}{2}} dt.$$

On the other hand, from (10) we know that

$$\tau'(t_B) = \tau'(t_A) \left[\frac{g(\gamma'(t_B), \gamma'(t_B))}{g(\gamma'(t_A), \gamma'(t_A))} \right]^{\frac{1}{2}} e^{-\frac{1}{2} \int_{t_A}^{t_B} \omega(\gamma'(s)) ds}.$$

After substituting the above equation into the expression for τ_{BC} we have

$$\tau_{BC} = \frac{\tau'(t_A)}{(-g(\gamma'(t_A), \gamma'(t_A)))^{\frac{1}{2}}} \int_{t_B}^{t_C} e^{-\frac{1}{2} \int_{t_A}^t \omega(\gamma'(s)) ds} (-g(\gamma'(t), \gamma'(t)))^{\frac{1}{2}} dt.$$

Finally, adding up the results we obtain

$$\begin{aligned} \tau_{AB} + \tau_{BC} &= \frac{\tau'(t_A)}{(-g(\gamma'(t_A), \gamma'(t_A)))^{\frac{1}{2}}} \int_{t_A}^{t_B} e^{-\frac{1}{2} \int_{t_A}^t \omega(\gamma'(s)) ds} (-g(\gamma'(t), \gamma'(t)))^{\frac{1}{2}} dt \\ &\quad + \frac{\tau'(t_A)}{(-g(\gamma'(t_A), \gamma'(t_A)))^{\frac{1}{2}}} \int_{t_B}^{t_C} e^{-\frac{1}{2} \int_{t_A}^t \omega(\gamma'(s)) ds} (-g(\gamma'(t), \gamma'(t)))^{\frac{1}{2}} dt \\ &= \frac{\tau'(t_A)}{(-g(\gamma'(t_A), \gamma'(t_A)))^{\frac{1}{2}}} \int_{t_A}^{t_C} e^{-\frac{1}{2} \int_{t_A}^t \omega(\gamma'(s)) ds} (-g(\gamma'(t), \gamma'(t)))^{\frac{1}{2}} dt \\ &= \tau_{AC} \end{aligned}$$

which proves the additivity of proper time intervals.

From the above arguments we are led to conclude that Perlick's definition of proper time is both mathematically consistent and physically sensible. In the next section, we will show that this definition is, in fact, equivalent to the definition proposed by Ehlers, Pirani and Schild in their original paper [1].

IV. EQUIVALENCE OF PERLICK'S AND EPS PROPER TIME.

Let us begin by recalling the definition of proper time given by EPS [1]: We say that a time-like curve γ is parametrized by proper time if the tangent vector γ' is *congruent* at each point of the curve to a non-null vector V which is parallel-transported along γ . In the previous sentence, *congruence* of vectors at a point means that both vectors have the same *norm*, i.e., $g_p(\gamma'_p, \gamma'_p) = g_p(V_p, V_p)$. We will now establish the equivalence between both definitions.

Proposition 2. *A time-like curve is parametrized by proper time according to EPS if it is parametrized by proper time according to definition 2.*

Proof. Suppose that we have a time-like curve γ parametrized by proper time according to EPS and let τ denote such parametrization. Then, by definition, there exists a parallel vector field V along γ satisfying the following condition:

$$g(\gamma'(\tau), \gamma'(\tau)) = g(V(\tau), V(\tau)).$$

Thus, differentiating the previous identity, using Weyl's compatibility condition and the fact that V is a parallel vector field, we get the following.

$$\begin{aligned} \omega(\gamma'(\tau))g(V(\tau), V(\tau)) &= 2g(\gamma'(\tau), \frac{D\gamma'(\tau)}{d\tau}) + \omega(\gamma'(\tau))g(\gamma'(\tau), \gamma'(\tau)) \\ &= 2g(\gamma'(\tau), \frac{D\gamma'(\tau)}{d\tau}) + \omega(\gamma'(\tau))g(V(\tau), V(\tau)) \Rightarrow \\ 0 &= g(\gamma'(\tau), \frac{D\gamma'(\tau)}{d\tau}) \end{aligned}$$

Hence γ is parametrized by proper time according to definition 2.

In order to prove the converse, suppose that γ is parametrized by proper time according to definition 2. Then, by hypothesis, $\gamma'(\tau)$ and $\frac{D\gamma'(\tau)}{d\tau}$ are orthogonal. Now consider the

following initial value problem:

$$\begin{aligned}\frac{DV(\tau)}{d\tau} &= 0, \\ V(\tau_0) &= \gamma'(\tau_0),\end{aligned}$$

which defines a unique parallel vector field along γ . Since V is a parallel vector, using Weyl's compatibility condition, we see that $g(V(\tau), V(\tau))$ satisfies the following equation

$$\frac{d}{d\tau}g(V(\tau), V(\tau)) = \omega(\gamma'(\tau))g(V(\tau), V(\tau)).$$

Using the compatibility condition and the fact that $\gamma'(\tau)$ and $\frac{D\gamma'(\tau)}{d\tau}$ are orthogonal, we see that $g(\gamma'(\tau), \gamma'(\tau))$ also satisfies the previous equation. Furthermore, both solutions initially agree, that is, $g(V(\tau_0), V(\tau_0)) = g(\gamma'(\tau_0), \gamma'(\tau_0))$. Then, by uniqueness of solutions, we get that $g(V(\tau), V(\tau)) = g(\gamma'(\tau), \gamma'(\tau))$, which means that γ is parametrized by proper time according to EPS. \square

With the previous result we have established the equivalence of both definitions. This is an interesting fact, since, at first sight, the two definitions do not seem to be intimately related to each other. It is also worth noting that these two definitions have been widely used in the literature. However, it seems to us, that when an author accepts one of them, makes no reference to the other ([1],[3],[4],[6],[9]). Thus, if we establish the equivalence between the two definitions, then there will be no ambiguity any longer.

V. ANALYSIS OF THE SECOND CLOCK EFFECT

In this section, we investigate the question of whether or not a space-time modelled as a Weyl structure, in which proper time is understood according to Perlick's definition, exhibits the so-called *second clock effect*. As is well known, we say that a space-time model exhibits the second clock effect if the clock rate of clocks depends on their histories [10].

In order to give an answer to this question consider the following situation. Suppose that we transport two identical standard clocks along a time-like curve from A to B , and then, at B , they separate, following different paths γ_1 and γ_2 until they merge again at event C (see Fig. 2), after which they continue their journey together along the same path. Suppose that both clocks were synchronized at A . Thinking of a clock as a device that counts the number

of cycles of some periodic process, what we are saying when we refer to synchronization is that identical clocks use the same type of process and that the periods of these cycles were set to be equal at A (both clocks are set with the same scale at A). Now assume that our space-time model does not exhibit a second clock effect. Accepting this hypothesis means that we would expect that the clock rate of a clock, at a given event in space-time, should depend only on *local properties* of the clock, that is, its position, instant velocity, instant acceleration, etc, but not on its history. Therefore, in the particular case we are considering, after we bring back the two identical clocks together at C , and keep them together, we would expect their clock rates to coincide. In other words, we would expect the number of cycles counted by either clock after C to be the same. This also means that the readings of the two clocks would coincide at any subsequent event D ($\tau_{CD} = \bar{\tau}_{CD}$). These considerations give us a way to test a possible existence of the second clock effect: compute the elapsed time for both clocks between C and some subsequent event D and see whether they agree or not. If they do not, then clearly there is a second clock effect.

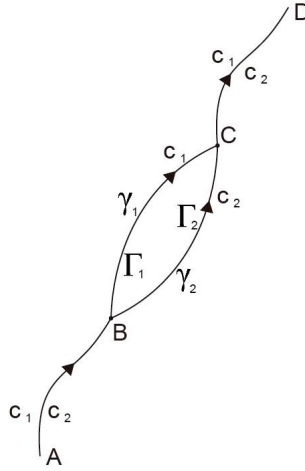


FIG. 2: Transport of two clocks c_1 and c_2 . The curves γ_1 and γ_2 represent the world lines of c_1 and c_2 respectively, while Γ_1 and Γ_2 denote the portions of these curves between B and C .

Let us now, with the help of (11), carry out some calculations to make the above discussion more precise. First, let us recall that after the event C both clocks are being transported along the same time-like curve γ . Suppose that the curve γ is parametrized by some arbitrary parameter u . Then, from (11) we have

$$\tau = \frac{\tau'(u_C)}{(-g(\gamma'(u_C), \gamma'(u_C)))^{\frac{1}{2}}} \int_{u_C}^u e^{-\frac{1}{2} \int_{u_C}^{u'} \omega(\gamma'(s)) ds} (-g(\gamma'(u'), \gamma'(u')))^{\frac{1}{2}} du'$$

$$\bar{\tau} = \frac{\bar{\tau}'(u_C)}{(-g(\gamma'(u_C), \gamma'(u_C)))^{\frac{1}{2}}} \int_{u_C}^u e^{-\frac{1}{2} \int_{u_C}^{u'} \omega(\gamma'(s)) ds} (-g(\gamma'(u'), \gamma'(u')))^{\frac{1}{2}} du'$$

where τ and $\bar{\tau}$ represent the readings of clocks 1 and 2 respectively. We can also compute τ_{AC} and $\bar{\tau}_{AC}$ using (11). In order to make this computation, we will consider that the parametrization u used for γ after C is the parametrization used to parametrize the whole time-like path of clock 1. On the other hand, for the path of clock 2 we will use \bar{u} as a parameter. Then, a straightforward calculation shows that

$$\tau'(u_C) = \tau'(u_A) \left(\frac{g(\gamma'_1(u_C), \gamma'_1(u_C))}{g(\gamma'_1(u_A), \gamma'_1(u_A))} \right)^{\frac{1}{2}} e^{-\frac{1}{2} \int_{u_A}^{u_C} \omega(\gamma'_1(s)) ds}$$

$$\bar{\tau}'(\bar{u}_C) = \bar{\tau}'(\bar{u}_A) \left(\frac{g(\gamma'_2(\bar{u}_C), \gamma'_2(\bar{u}_C))}{g(\gamma'_2(\bar{u}_A), \gamma'_2(\bar{u}_A))} \right)^{\frac{1}{2}} e^{-\frac{1}{2} \int_{\bar{u}_A}^{\bar{u}_C} \omega(\gamma'_2(s)) ds}$$

where γ_1 and γ_2 are the curves representing the world lines of each clock. Now, since from A to B and after C both world lines are the same, both curves being equal in these intervals, we could reparametrize γ_2 by changing from \bar{u} to u to obtain

$$\bar{\tau}'(\bar{u}_C) = \bar{\tau}'(u_A) \frac{d\bar{u}(\bar{u}_C)}{du} \left(\frac{g(\gamma'_2(u_C), \gamma'_2(u_C))}{g(\gamma'_2(u_A), \gamma'_2(u_A))} \right)^{\frac{1}{2}} e^{-\frac{1}{2} \int_{\bar{u}_A}^{\bar{u}_C} \omega(\gamma'_2(s)) ds}.$$

Thus we have

$$\bar{\tau}'(u_C) = \frac{d\bar{u}(u_C)}{du} \bar{\tau}'(\bar{u}_C) = \bar{\tau}'(u_A) \left(\frac{g(\gamma'_2(u_C), \gamma'_2(u_C))}{g(\gamma'_2(u_A), \gamma'_2(u_A))} \right)^{\frac{1}{2}} e^{-\frac{1}{2} \int_{\bar{u}_A}^{\bar{u}_C} \omega(\gamma'_2(s)) ds}.$$

Since $\gamma'_1(u_A) = \gamma'_2(u_A)$ and $\gamma'_1(u_C) = \gamma'_2(u_C) = \gamma'(u_C)$ (recalling that it is the same curve with the same parametrization), then for the reading of clock 2 we get

$$\bar{\tau} = \bar{\tau}'(u_A) \left(\frac{g(\gamma'_1(u_C), \gamma'_1(u_C))}{g(\gamma'_1(u_A), \gamma'_1(u_A))} \right)^{\frac{1}{2}} e^{-\frac{1}{2} \int_{\bar{u}_A}^{\bar{u}_C} \omega(\gamma'_2(s)) ds} \frac{\int_{u_C}^u e^{-\frac{1}{2} \int_{u_C}^{u'} \omega(\gamma'(s)) ds} (-g(\gamma'(u'), \gamma'(u')))^{\frac{1}{2}} du'}{(-g(\gamma'(u_C), \gamma'(u_C)))^{\frac{1}{2}}}$$

$$= \frac{\bar{\tau}'(u_A)}{\tau'(u_A)} e^{\frac{1}{2} \int_{u_A}^{u_C} \omega(\gamma'_1(s)) ds - \frac{1}{2} \int_{\bar{u}_A}^{\bar{u}_C} \omega(\gamma'_2(s)) ds} \tau.$$

Also, as both clocks have the same scale at the event A , that is,

$$\frac{\bar{\tau}'(u_A)}{\tau'(u_A)} = \frac{d\bar{\tau}(\tau_A)}{d\tau} = 1,$$

we finally get

$$\bar{\tau} = e^{\frac{1}{2} \int_{\Gamma_1} \omega - \frac{1}{2} \int_{\Gamma_2} \omega} \tau. \quad (14)$$

Thus we are led to conclude that

A Weyl space-time does not exhibit a second clock effect if, and only if, $\int_{\Gamma_1} \omega = \int_{\Gamma_2} \omega$, where γ_1 and γ_2 are time-like curves joining the same pair of events A and B .

Taking into consideration the previous statement, we are now in position to discuss whether or not a Weyl structure is a suitable model for space-time. In this framework we have an unambiguous, well-defined and physically sensible definition for proper time, and we have arrived at the conclusion that a general Weyl space-time exhibits a second clock effect. We note that up to now this kind of effect has never been measured, and we could also give interesting arguments, which would, at least, set strong constraints on the values of this effect (see, for example, [11]). Then, it is natural to ask, what would be the most general mathematical structure of space-time if we were to reject space-times exhibiting a second clock effect. If, in the previous statement, we could drop the condition that the curves γ_1 and γ_2 are time-like, then it is immediate to see that the non-existence of a second clock effect would imply that the 1-form ω must be closed. If, in turn, we assume that space-time is simply connected, then we would be led to a Weyl integrable space-time. In what follows, we will show that even if the timelike character of the curves is kept, a Weyl integrable space-time still emerges as a consequence of the non-existence of a second clock effect. Thus, let us suppose that ω is path independent when integrated over time-like curves and see what this implies. First consider an arbitrary event in space-time, $p \in M$, and the set of events I_p that are causally connected with p . Thus I_p is defined as

$$I_p \doteq I_p^+ \bigcup I_p^- = \{q \in M, \text{ such that there is a time-like curve joining } q \text{ and } p\},$$

which is an open subset of M . On this set define the following function:

$$f(q) \doteq \int_{\gamma} \omega$$

where γ is any time-like curve joining p and q (see figure 3) Since, by hypothesis, the integral of ω over any such time-like curve does not depend on the choice of the curve, then the previous function is well-defined.

Before going any further, it is important to remark that f is differentiable near p . To see this, we explicitly compute f in a normal coordinate system around p . In such a coordinate system computing f is not difficult since we can choose as the time-like curve γ joining p and q , with q sufficiently near to p , the unique time-like geodesic joining these points inside the normal neighborhood. Then, since the coordinate expression of a geodesic in a normal

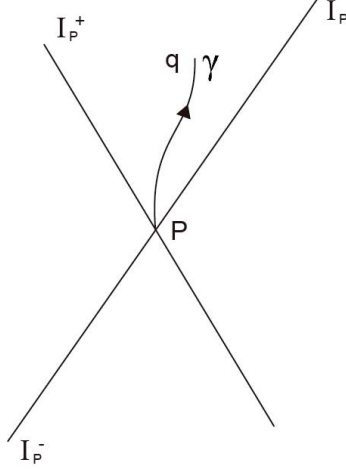


FIG. 3: Domain of f .

coordinate system is given by a straight line in the direction of the initial velocity of the geodesic (see, for example, [12]), we can make an easy explicit computation for f . Doing this, we get the following:

$$f(x) = x^\alpha \int_0^1 \omega_\alpha(tx) dt. \quad (15)$$

Assuming ω to be smooth, it is clear that f is differentiable near p . From now on we will restrict f to such a neighborhood of p , so that we know that df exists. Consider now a neighborhood U_q of q such that $U_q \subset I_p$ and a time-like vector $X_q \in T_q M$. Then, we can construct a smooth curve μ defined in a neighborhood J_q of the origin satisfying

$$\begin{aligned} \mu(0) &= q \\ \mu'(0) &= X_q \end{aligned}$$

Also, since $g_q(X(q), X(q)) < 0$, then, by continuity, there is a neighborhood of $0 \in \mathbb{R}$ where μ is time-like. To avoid complications in the notation we will take J_q to be such a neighborhood (if needed, we could shrink the original J_q in order for this to be true). Now we wish to compute

$$df_q(X_q) = \left. \frac{df \circ \mu(t)}{dt} \right|_{t=0}.$$

We can compute $f \circ \mu(t)$ in the following way. First consider a piecewise smooth time-like curve joining p and $\mu(t)$ consisting of a time-like curve β , joining p and q , and then the curve

μ joining q and $\mu(t)$. In this set-up we have the following:

$$\begin{aligned} f \circ \mu(t) &= \int_{\beta} \omega + \int_{\mu} \omega \\ &= f(q) + \int_o^t \omega(\mu'(u)) du. \end{aligned}$$

Therefore

$$\begin{aligned} \frac{df \circ \mu(t)}{dt} \Big|_{t=0} &= \omega_{\mu(0)}(\mu'(0)) \\ &= \omega_q(X_q). \end{aligned}$$

Thus we have shown that $df_q(X_q) = \omega_q(X_q)$, for any time-like $X_q \in T_qM$.

$$df_q(X_q) = \omega_q(X_q) \quad \forall X_q \in T_qM \text{ time like.}$$

To see how df_q acts on an arbitrary vector of T_qM consider an orthonormal basis $\{e_0, e_i\}$ of T_qM , where e_0 is time-like and e_i are space-like. Now if we consider the set of vectors in T_qM given by

$$\begin{aligned} \tilde{e}_0 &\doteq e_0 \\ \tilde{e}_i &\doteq 2e_0 + e_i \end{aligned}$$

then $\{\tilde{e}_0, \tilde{e}_i\}$ gives a basis of time-like vectors for T_qM . Then if we pick $V \in T_qM$ arbitrary and we write it in this basis $V = V^\alpha \tilde{e}_\alpha$, we can then compute the action of df_q on this element:

$$\begin{aligned} df_q(V) &= V^\alpha df_q(\tilde{e}_\alpha) \\ &= V^\alpha \omega_q(\tilde{e}_\alpha) \\ &= \omega_q(V) \Rightarrow \\ df_q(V) &= \omega_q(V) \quad \forall V \in T_qM \text{ and } q \in I_p. \end{aligned}$$

Therefore we have shown that, given any point $q \in I_p$, there is a neighborhood of q where ω is exact. Then, if we accept that any event q in space-time lies in I_p for some other event p , we conclude that ω is closed. Finally, assuming that space-time is simply connected, then, to avoid the second clock effect, our space-time model has to be reduced to a Weyl integrable structure $(M, [g], [\phi])$.

VI. FINAL COMMENTS

In this paper we have revisited the notion of proper time in the framework of Weyl's space-time within the axiomatic approach put forward by Ehlers, Pirani and Schild. We have shown that the EPS original definition of proper time is equivalent to the definition proposed by V. Perlick, even though the latter seems to be more easily motivated in the context of EPS's axiomatic approach to space-time. After showing that Perlick's notion leads to a well-defined and physically sensible definition of proper time in a general Weyl space-time, without invoking any additional axioms, we proved that this kind of space-time exhibits a second clock effect. We then derived the condition a space-time must obey in order that a second clock effect does not appear. We have shown that in this case the geometric space-time structure should be that of a Weyl integrable space-time. Our final conclusion is that, within the slightly modified EPS axiomatic approach, taking into account Perlick's proper time, Weyl integrable space-time appears naturally as the most general model for space-time. However, the question of a possible existence of a second clock effect, which would then widen this scenario leaving us with a general Weyl space-time as a suitable mathematical model, is something that should be settled by experiment.

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